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Lecture 6

Active Filters

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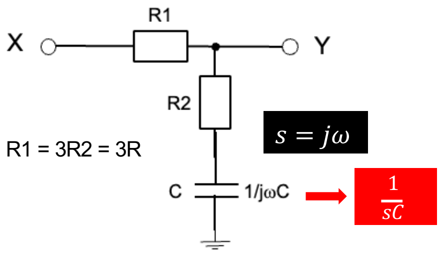
In this lecture, we will explore how to use op-amps to implement active filters. Again, part of the contents of this lecture will be explored during Laboratory Experiment 2 this week and next week.

Transfer Function of 1st order LP Filter

From Year 1 ADC Part 1 Lecture 11, slide 3.



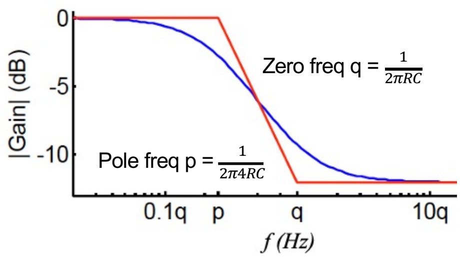
- More general if use **complex frequency s** to represent the quantity $j\omega$.
- Covered in Signals and Systems module this term, and Control Systems next term.
- Express impedance of capacitor as $\frac{1}{sC}$ instead of $\frac{1}{j\omega C}$.
- Capture both steady state (ac) and transient behaviour .
- Transfer function defined as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{R+1/sC}{4R+1/sC} = \frac{1+sRC}{1+4sRC}.$$

- Frequency response is calculated as

$$H(s)|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)} = \frac{1+j\omega RC}{1+4j\omega RC}.$$

- Easier to perform algebra manipulation than using $j\omega$.
- Provides better intuitions on system behaviour.
- This simple filter is first-order low-pass with 1 pole and 1 zero.



- The break frequency occurs when real and imaginary parts are equal in numerator (zero freq) and denominator (pole freq).

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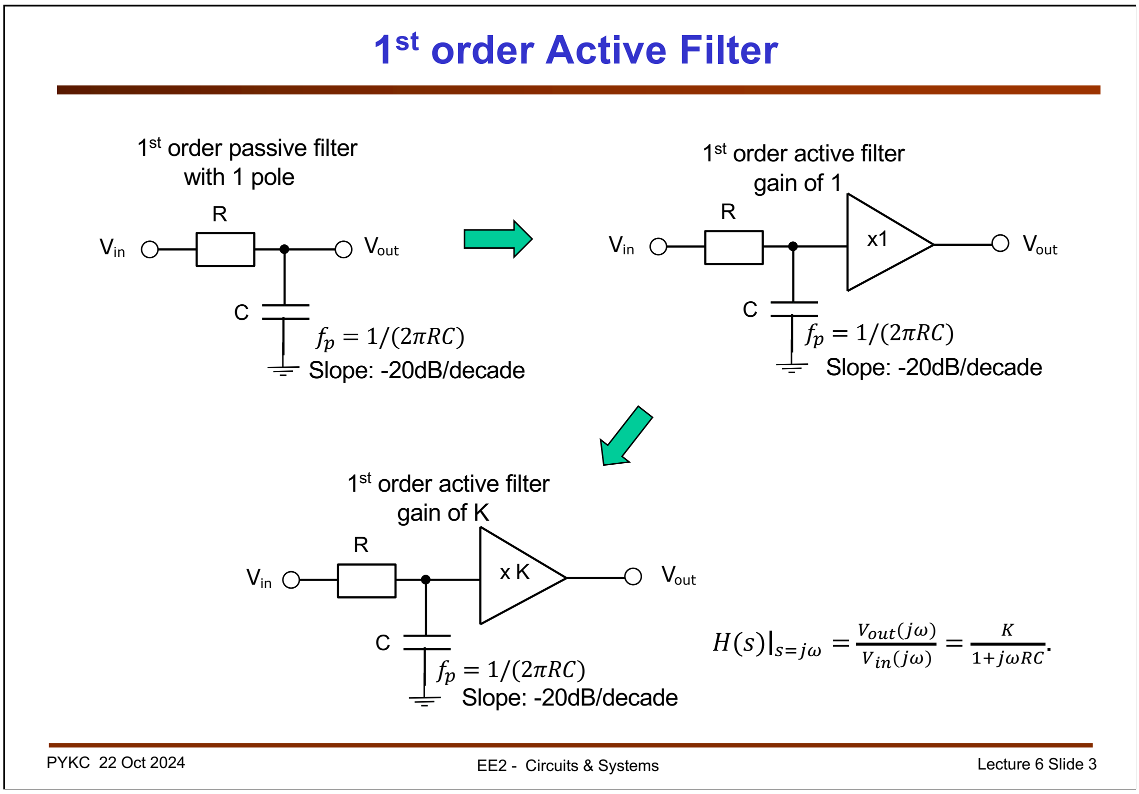
Consider the simple RC network shown. This was analysed in Year 1 ADC Lecture 11 (slide 3) already. However, in this case, we substitute s for jω. The variable s is known as "**complex frequency**". You will learn about this in two other modules: Signals and Linear Systems, and in Control Engineering.

The reason s is used in preference to jω because frequency response H(jω) is only valid if the circuit (or system) is in steady state (i.e. all transients have died down) and all signals are expressed as sine waves. Using complex frequency s allows both transient and steady state behaviours to be analysed.

The impedance of a capacitor is 1/sC instead of 1/jωC.

Furthermore, you will learn in other modules the relationship between the transfer function H(s) expressed as products of factors in s, and how this relates to the idea of poles and zeroes. (This is outside the scope of this module.)



Simple RC network as a low pass filter suffers from loading effect – as soon as a load is applied to the output, the filter characteristic changes.

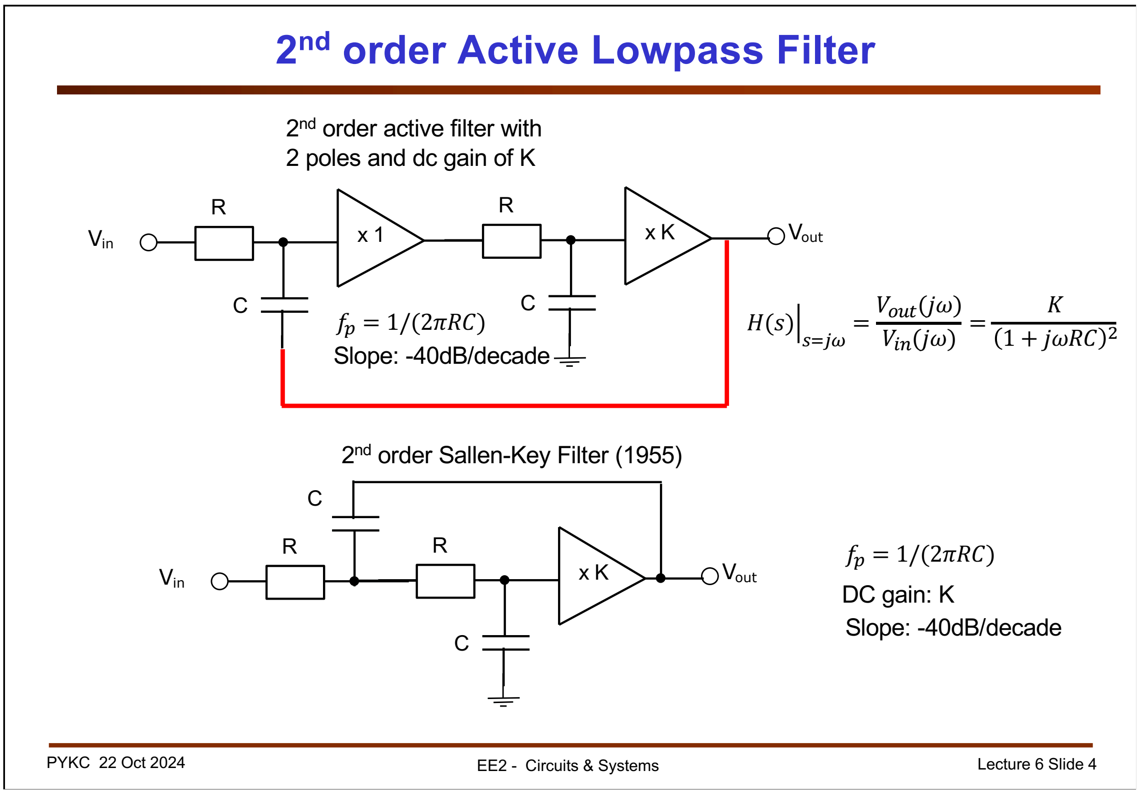
One can simple add a unity gain buffer (x1 amplifier) to isolate the output from the input. This is a simple 1st order active filter. It is called an active filter because this circuit requires an amplifier with 'active' component (i.e. transistors and amplification).

One R and one C resulted in a 1st order circuit with one pole (a concept to be covered next term in the Control Engineering module). The pole frequency f_p of the active filter is now NOT affected by the load (up to the op-amp output current limits). It is given by $f_p = \frac{1}{2\pi RC}$. At this frequency, the gain of the filter is -3dB.

Instead of using a unity gain amplifier, we can use a non-inverting amplifier with a gain of K as shown.

In all three cases, the low pass filter has an attenuation that falls off at a rate of -20dB per decade.

This filter is first order because it has only one reactive component (the capacitor). A second order filter will have TWO capacitors, and the fall off rate will be -2 x 20dB per decade. Generally, an nth order filter has a fall off rate of – n x 20dB/decade.



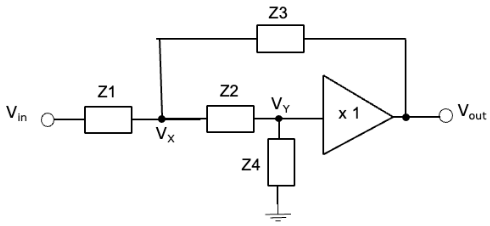
Cascading two identical 1st order filters gives a 2nd order filter. The pole frequency remains the same as before, but the attenuation at this frequency is now -6dB instead of -3dB. The fall off rate is, as expected, -40dB/decade.

Instead of using two op-amps, one can connect the ground of the left capacitor to the output of the 2nd op-amp as shown above. This feedback arrangement eliminates the need of the unity gain buffer amplifier and implements a 2nd order low pass filter using two capacitors, two resistors, and an amplifier with gain of K.

K can be set to 1 but must be lower than 3 (otherwise the circuit will become unstable).

This circuit arrangement (called topology) is known as a Sallen-Key filter.

Sallen-Key Filter Topology



- Invented by R.P. Sallen and E.L. Key in 1955 using valves as active devices (!)
- Z1 to Z4 are arbitrary impedance from resistors, capacitors or inductors.
- Assume amplifier gain is 1 (can be generalised to K), $V_Y = V_{out}$.
- Apply KCL to V_X yields:

$$\frac{V_{in} - V_x}{Z_1} + \frac{V_{out} - V_x}{Z_3} + \frac{V_{out} - V_x}{Z_4} = 0$$

- Apply KCL at V_Y yields:

$$V_x = V_{out} + \frac{Z_2}{Z_4} V_{out} = V_{out}(1 + \frac{Z_2}{Z_4})$$

- Combining the two gives a general transfer function equation:

$$\frac{V_{out}}{V_{in}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1 + Z_2) + Z_3 Z_4}$$

The circuit shown in this slide is the general form of Sallen and Key filter published in 1955. The passive components are replaced by generic impedances Z1 to Z4.

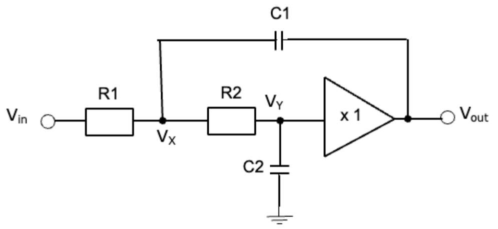
For simplicity, we assume K=1.

Z1 to Z4 can be resistors, capacitors or inductors, although normally only resistors and capacitors are used because inductors are not easy to manufacture in integrated circuits, and their performance are generally not high enough for implementing good filter circuits.

To derive the transfer function of this circuit (i.e. V_{out}/V_{in}), we can apply KCL to the circuit nodes X and Y as shown in the slide.

By choosing whether to use resistor or capacitor for the components in this circuit topology, one can implement low pass, high pass, band pass and band stop filters. The Sallen-Key filter is therefore general and is one of most widely used active filter in electronic circuits.

2^{nd} order Sallen-Key Lowpass Filter



$$\frac{V_{out}}{V_{in}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1 + Z_2) + Z_3 Z_4}$$

$Z_1 = R_1, \ Z_2 = R_2$

$Z_3 = 1/sC_1 \ , \ Z_4 = 1/sC_2$

- Using the transfer function equation $H(s)$ from previous slice:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{s^2 C_1 C_2}}{R_1 R_2 + \frac{1}{sC_1}(R_1 + R_2) + \frac{1}{s^2 C_1 C_2}}$$

- Rearrange and put this in a standard form for a 2^{nd} order lowpass system:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + \boldsymbol{C_2(R_1 + R_2)}\, s + \boldsymbol{C_1 C_2 R_1 R_2}\, s^2}$$

Consider the above circuit topology with R1, R2 and C1, C2. Assuming that K = 1 for simplicity. We can substitute these component values to Z1 to Z4, and derive the above transfer function.

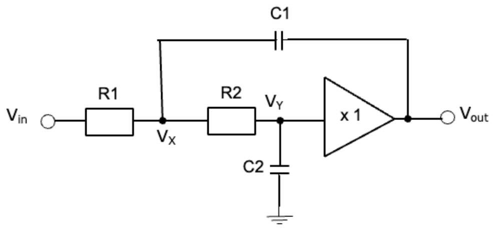
Note that we are using the s-notation to denote the impedance of capacitor as 1/sC. You have learned Laplace transform in Year 1 mathematics. You will also be learning about systems theory where s is used as the Laplace transform variable. The general idea is that $s = \alpha + j\omega$, and is often referred to as 'complex frequency'.

Unlike Fourier transform that only uses $j\omega$ and is only applicable to steady state conditions, using complex frequency allows us to model systems for both transient and steady state behaviour. Further, using s and Laplace transform, we convert differential equations into algebra equations, which makes analysis easier.

The ratio of output to input in the s-domain is known as transfer function H(s).

In this case, we obtain the transfer function of this Salley-Key low pass filter circuit into the form shown above with denominator with s to the power of 2 – it is therefore a 2^{nd} order system equation.

Significance of ω_0 and Q (1)



$Z_1 = R_1, \quad Z_2 = R_2$
$Z_3 = 1/sC_1 \;, \quad Z_4 = 1/sC_2$

- Put this into the standard form of a 2nd order lowpass filter is:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+\frac{1}{\omega_0 Q}s+\frac{1}{\omega_0^2}s^2} = \frac{1}{1+C_2(R_1+R_2)\,s+C_1C_2R_1R_2\,s^2}$$

Q is the quality factor
ζ is the damping ratio
$Q = \frac{1}{2\zeta}$

$\frac{2\zeta}{\omega_0} = \frac{1}{\omega_0 Q}$

$\frac{1}{\omega_0^2}$

ω_0 is the cutoff frequency (rad/s)
$f_c = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{C_1C_2R_1R_2}}$ in Hz

- Therefore, the cutoff frequency is: $f_c = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{C_1C_2R_1R_2}}$ Hz
- The quality factor Q is: $Q = \frac{\sqrt{C_1C_2R_1R_2}}{C_2(R_1+R_2)}$

Let us now examine the significance of the coefficient (constant values) associated with the s and s² terms in the denominator.

The coefficient of the s² term is $C_1C_2R_1R_2$, and it defines the break or cutoff frequency ω_0 of the filter.

The coefficient of the s term is $C_2(R_1 + R_2)$, and it specifies (given ω_0) the quality factor Q of the filter, which is directly related to the damping factor ζ (pronounced as 'zeta') of the system.

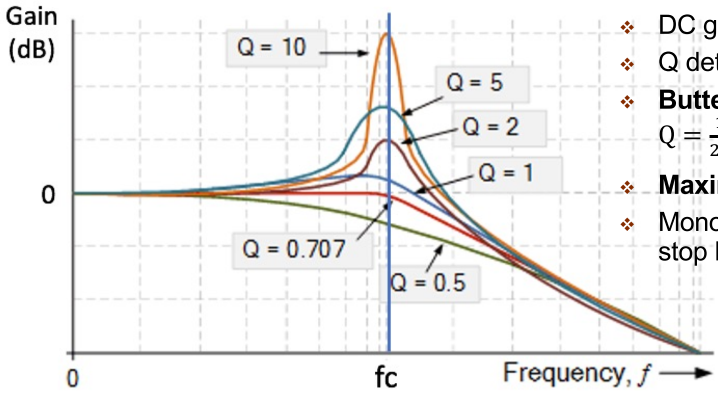
In filter circuits, we use Q instead of ζ. In control theory, we use ζ instead of Q.

In general, a filter with a higher Q factor means that it is more 'selective' and has a more 'peaky' characteristics. Higher Q also means the circuit tends to be more oscillatory.

Significance of ω_0 and Q (2)

- Rewrite the transfer function as:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + \frac{1}{\omega_0 Q}s + \frac{1}{\omega_0^2}s^2} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad f_0 = \frac{\omega_0}{2\pi}$$



- ω_0 is the **cut-off frequency** of filter.
- DC gain of filter is 1 (i.e. s=0).
- Q determine how ‘peaky’ the filter is.
- **Butterworth filter**: $2\zeta = 1.414$, $\mathrm{Q} = \frac{1}{2\zeta} = 0.707$.
- **Maximally flat** gain in passband
- Monotonically decreasing gain in stop band.

We can re-arrange the transfer function as shown above to demonstrates the significance of ω_0 and Q.

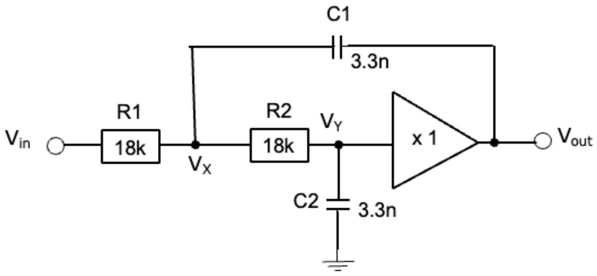
ω_0 defines the frequency at which the low pass filter starts attenuating the input signal. However, Q governs how peaky the filter response is at the cutoff frequency. If Q < 0.707, the filter drop off early and there is no resonance at ω_0.

If Q > 0.707, the filter characteristics starts to show higher than x1 gain at ω_0. The higher the Q value, higher the gain at this frequency.

Q = 0.707 is a special case when the filter is known as ‘maximally flat’ meaning that the gain stays at 0dB as long as possible before falling off, but it never rises above 0dB. Such maximally flat filter is known as a Butterworth filter.

In the case of Q = 0.5, the gain also never goes above 0dB, but the fall off starts much earlier and therefore the gain characteristic is not as flat as the maximally flat case.

A simple Sallen-Key filter (from Lab 2)



- Simplify by making $R_1 = R_2 = R = 18k\Omega$, and $C_1 = C_2 = C = 3.3nF$.
- The cutoff frequency is: $f_c = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}} = \frac{1}{2\pi RC}$ = 2.7kHz and
- $Q = \frac{\sqrt{R_1R_2C_1C_2}}{C_2(R_1+R_2)} = \frac{RC}{(C\times 2R)} = \frac{1}{2}$.
- This is NOT a Butterworth filter because Q is not 0.707 or $\frac{1}{\sqrt{2}}$.

Let us consider a concrete Sallen-Key filter as used in Lab Experiment 2.

Here, R1=R2=R=18kΩ. C1=C2=C=3.3nF.

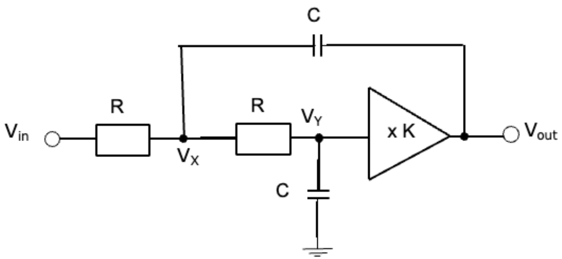
The transfer function now simplifies significantly resulting in simple equation for calcuting the break (or cutoff) frequency:

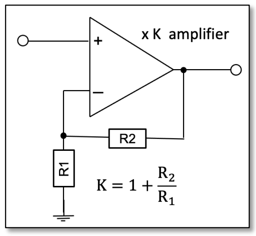
$$f_c = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}} = \frac{1}{2\pi RC} = 2.7\text{kHz}$$

However, this filter has a Q value of 0.5. This is therefore not a Butterworth filter. Note that in this particular case, Q is NOT dependent of RC, which is fixed by the cutoff frequency!

To specify a Q value different from 2, one can either use different values for C1 and C2, or introduce a gain to the op-amp such that the gain is 1 < K < 3,

Sallen-Key filter with gain = K





- Keep same R and C values, fix Q by changing gain of op-amp K
- Left as an exercise to proof:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = K \times \frac{1}{1 + \underbrace{(3-K)RC}_{\frac{1}{\omega_0 Q}}\, s + \underbrace{R^2C^2}_{\frac{1}{\omega_0^2}}\, s^2}$$

- Therefore, cutoff frequency f_c is same as before: $f_c = \frac{1}{2\pi\omega_0} = \frac{1}{2\pi RC}$.
- And, $Q = \frac{1}{\omega_0} \times \frac{1}{(3-K)RC} = \frac{1}{3-K}$.
- Therefore, to get a Butterworth filter with this topology, Q = 0.707, and
- $K = 3 - \frac{1}{Q} = 1.586.$

Consider the above circuit, which is identical to the circuit before except that the amplifier has gain of K instead of 1.

It is left as a tutorial exercise for you to derive the transfer function, which is shown above.

The cutoff frequency, which is dependent only on RC product, it is therefore unchanged as the previous circuit.

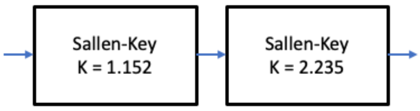
However, Q can now be changed with K. For Butterworth filter, Q is 0.707, and K is derived to be 1.586.

The xK amplifier can easily be implemented using an op-amp in the non-inverting amplifier configuration as shown in the slie.

General Procedure: Butterworth LP filter

1. Determine the required cutoff frequency f_c.
2. Calculate R and C product with: $RC = \frac{1}{2\pi f_c}$.
3. Pick a suitable value of C >> input capacitance of op-amp (say in nF range).
4. Calculate value of R to give the required cutoff frequency.
5. Determine order of filter depending on required attenuation rate. Filter attenuation rate is -20 x n dB/decade, for an nth order filter.
6. Round n to the nearest high even number. You will need n/2 Salley-Key filter stages.
7. Use the table below to design gain of each stage of the filter. For example, for a 4th order Butterworth filter, we need two Sallen-Key stages with gain of 1.152 followed by 2.235.
8. Choose resistors for op-amps feedback paths to provides specified gain values.

4th order Butterworth LP Filter



ORDER n	Gain values K
2	1.586
4	1.152, 2.235
6	1.068, 1.586, 2.483
8	1.038, 1.337, 1.889, 2.610

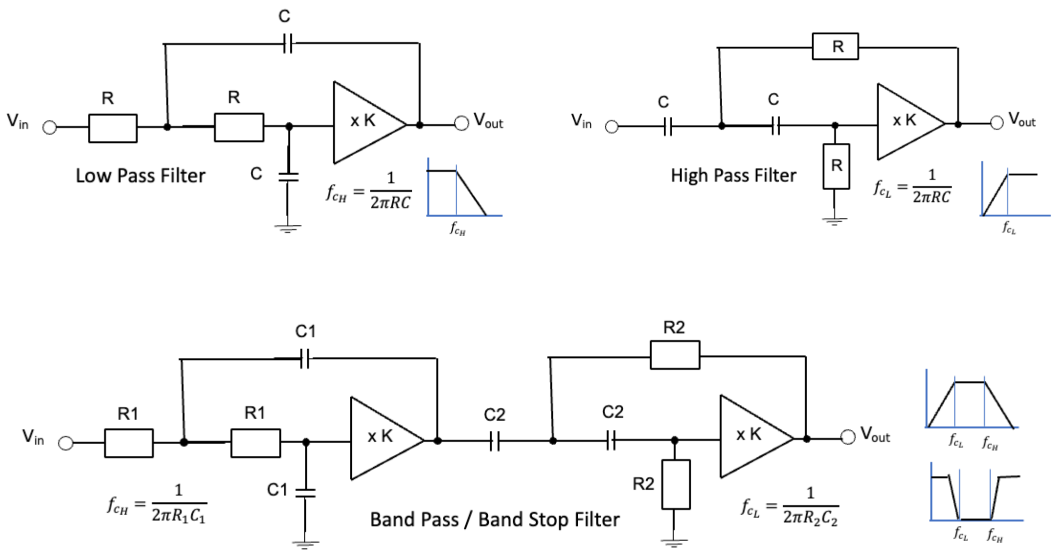
This slide shows the steps in designing a Butterworth low pass filter using the Sallen-Key topology with identical R and C the circuits.

Note that one must use the specified K values in the different stages of the filter.

Since each Sallen-Key stage implements a 2nd order responses, there is no real reason to have odd order of filter. For example, if one needs a 3rd order filter, one would need 2 op-amps in any case. Therefore one would use a 4th order filter with same cutoff frequency but steeper attenuation.

Finally, it is IMPORTANT TO NOTE that implementing filters using this circuit topology has one disadvantage. The DC gain of such a filter is NOT 0dB or x1. Instead, the 2nd order filter has a DC gain of 1.586 or +4dB. However, the frequency response is FLAT.

Other Sallen-Key filter circuits



To implement a high pass filter, capacitors are swapped into resistors, and vice versa.

To implement a band pass or band stop filter, cascade a low pass filter with a high pass filter and design the break frequencies accordingly.

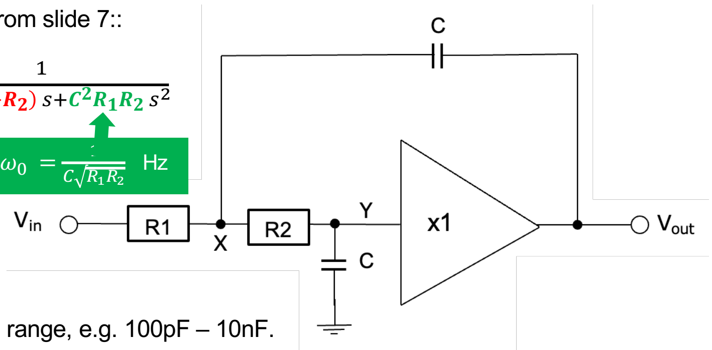
Using different values for R1 and R2

- Revisit transfer function of filter from slide 7::

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+\frac{1}{\omega_0 Q}s+\frac{1}{\omega_0^2}s^2} = \frac{1}{1+C(R_1+R_2)\,s+C^2R_1R_2\,s^2}$$

$$Q = \frac{\sqrt{R_1R_2}}{R_1+R_2}$$

$$\omega_0 = \frac{1}{C\sqrt{R_1R_2}} \text{ Hz}$$



- Design step:
  - Choose C in a reasonable range, e.g. 100pF – 10nF.
- Write down eq.1 for a given cutoff frequency.
- Write down eq. 2 for a given Q value.
- Solve the two equations for R1 and R2 values.

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So far, we have designed the Sallen-Key filter using same values for R and C. We then use RC product to determine the cutoff frequency and K to determine the Q factor.

This design method works well and is easy to follow, but has the disadvantage of resulting in a DC gain higher than 1.

If ensure that the gain is x1 or 0dB at DC, we can go back to using x1 unity gain amplifier. However, we now use two different values for R1 and R2. In this way, we are still able to design for any ω_o and Q values.